JANUARY

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

1
How many unique rectangles with integral sides can be formed with area of 2013 square units?

2
A limestone cliff is 20 m high. If calcium carbonate sediment accumulates at the rate of 1 cm/1000 years, how many years did it take for the cliff to reach that height? (The mineral calcium carbonate is the primary component of limestone.)

3
A fair six-sided die is rolled twice. What is the probability that the outcome of the second roll is greater than the outcome of the first roll?

4
A fair six-sided die is rolled three times. If the first, second, and third outcomes are denoted a, b, and c, respectively, what is the probability that $a < b < c$?

5
Use each integer from 1 to 9, inclusive, exactly once to write a numerical expression equal to 2013. An integer may be written as an exponent or as the radicand of a square root. The operations of addition, subtraction, multiplication, and division as well as grouping symbols may be used.

6
Scientists estimate that the flow southward of the North Atlantic deep-water current is 10 sverdrups. (A sverdrup is 1 million m^3/sec.) If the volume of the Atlantic Ocean is $3.24 \cdot 10^{17}$ m^3, how many years would be required for the entire Atlantic Ocean to circulate through deep water?

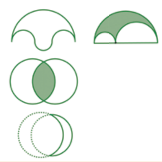
7
Assume that the oceans are at least $3 \cdot 10^9$ years old and that the mixing rate (see January 6 problem for the mixing rate) has been constant. How many times has the Atlantic Ocean been stirred through deep water?

8
The first two terms of a sequence are $a_1 = 4$ and $a_2 = 2$. Each subsequent term is defined by the rule $a_n = a_{n-1} - a_{n-2}$. What is the value of a_{2013}?

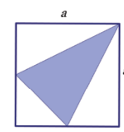
9
Let $\boxed{k}$ = the smallest prime number greater than k. If p_1, p_2, and p_3 are the distinct prime factors of 2013, compute the following:

$$\boxed{p_1} + \boxed{p_2} + \boxed{p_3} - \boxed{p_1 + p_2 + p_3}$$

10
Plane figures formed by intersecting arcs have some unusual names. Match the name with the figure: *arbelos* (shoemaker's knife), *lune* (Luna, Roman goddess of the moon), *salinon* (salt cellar), and *vesica piscis* (fish bladder).

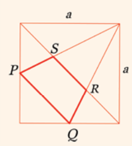


11
In a square with side length a, the midpoints of two adjacent sides are connected to each other and to the square's farthest vertex to form a triangle. Find the area of this triangle in terms of a.



12
Consider the isosceles triangle whose area was found in the January 11 problem. Find the length of the altitude to the base of that triangle in terms of a.

13
P and Q are the midpoints of the sides of a square with side length a. Find the area of quadrilateral $PQRS$ in terms of a.



14
Mathematician Charles L. Dodgson (Lewis Carroll) died on this date in 1898. His book *A Tangled Tale* opens with the following "knot": "Two travelers spend from 3 o'clock [p.m.] till 9 [p.m.] in walking along a level road, up a hill, and home again: their pace on the level being 4 miles an hour, up hill 3, and down hill 6. Find distance walked."

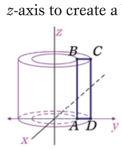
15
The three-digit numbers

acb, $a79$, $b0c$, and $bb1$

are consecutive terms in an arithmetic sequence. What digits are represented by a, b, and c?

16
Penny emptied her coin purse. She had 27 coins—pennies, nickels, dimes, and quarters—totaling \$3.30. If the pennies were quarters and the quarters were pennies, she would have only \$2.34. How much money would she have if the nickels were dimes and the dimes were nickels? (Pennies remain pennies, and quarters remain quarters.)

17
Rectangle $ABCD$ lies in the y,z-plane with A (0, 10, 0), B (0, 10, 25), and D (0, k, 0); $k > 10$. The rectangle is rotated about the z-axis to create a hollow cylinder. The rectangle is also rotated about the y-axis to create a solid cylinder. If the volumes of the two solids are equal, what is the value of k?



18
A Chinese text (ca. 200 BCE) poses this puzzle: A fox, a wild cat, and a dog go through a customs post; they are taxed 111 coins. The dog says to the wild cat, and the wild cat says to the fox, "Your skin is worth twice mine; you should pay twice as much tax!" If the 111 coins have the same value, how much does each pay to pass customs?

19
What is the smallest integer greater than 1 that is a perfect square, a perfect cube, a perfect fourth power, a perfect fifth power, and a perfect sixth power?

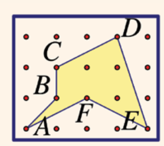
20
To write numbers in a vigesimal (base 20) system, we need 10 additional symbols. Let A = 10, B = 11, …, and so on up to J = 19. Complete the following equation, written in vigesimal notation. Each blank represents a single symbol.

3D • __ __ = 2 __ • __ __ = 2790

21
What is the largest possible remainder when a two-digit number is divided by the sum of its digits?

22
A *semiprime* is a composite number that is the product of two (possibly equal) primes. If the factors are not equal, the numbers are *square-free semiprimes*. Some square-free semiprimes are $2 \cdot 3 = 6$, $2 \cdot 5 = 10$, and $2 \cdot 7 = 14$. Find the smallest set of three consecutive integers that are square-free semiprimes.

23
In how many ways can a vertex be moved—1 unit at a time—to increase the polygon's area by 1 square unit?



24
Find the smallest positive integer such that the product of its digits is 120.

25
Joseph-Louis Lagrange was born on this date in 1736. He was the first to prove, in 1770, that every positive integer is the sum of at most four squares. Write the integer 2013 as the sum of four squares.

26
Two identical hexahedral dice have the triangular numbers 6, 10, 15, 21, 28, and 36, with one number on each face. If the two dice are tossed once, what is the probability that the sum of the faces is a square number?

27
Charles Dodgson was born on this date in 1832. The eighth "knot" in his *Tangled Tale* poses the following problem: "An oblong garden, half a yard longer than wide, consists entirely of a gravel-walk, spirally arranged, a yard wide and 3,630 yards long. Find the dimensions of the garden."

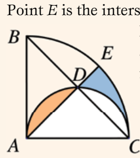
28
Given the array of letters shown, find the number of ways that the sequence

C-A-R-R-O-L-L

can be obtained if consecutive letters must be adjacent and if horizontal, vertical, and diagonal moves are allowed?

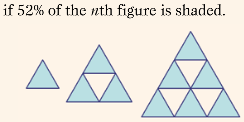
O L L L L L O
L R R R R R L
L R A A A R L
L R A C A R L
L R A A A R L
L R R R R R L
O L L L L L O

29
Segments AB and AC are of equal length, and $m\angle BAC = 90°$. Point D is the intersection of chord BC and the semicircle on $\overline{AC}$. Point E is the intersection of $\overset{\frown}{BC}$ and $\overline{AD}$. Find the ratio of the areas of the shaded regions.



30
When two parallel chords with lengths 40 and 48 lie on the same side of a circle's center, the distance between them is 8 units. What is the distance between them when they lie on opposite sides of the circle's center?

31
Each figure in the sequence consists of congruent triangles. As the number of interior triangles increases, the fraction of the figure that is shaded decreases. Find n if 52% of the nth figure is shaded.



SOLUTIONS to calendar

JANUARY

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The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

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1. Four. The factors of 2013 are 1, 3, 11, 33, 61, 183, 671, and 2013, so we have four pairs of factors that have a product of 2013.

2. Two million years. We know that 1 m = 100 cm, so 20 m = 2000 cm. Then

$$\frac{1\text{ cm}}{1000\text{ years}} = \frac{2000\text{ cm}}{y\text{ years}} \rightarrow y = 2{,}000{,}000,$$

or two million years.

3. 5/12. There are 6 • 6 = 36 possible outcomes of the two rolls. Of these, six are doubles—(1, 1), (2, 2), and so on—which do not satisfy the requirement that the outcome of the second roll must exceed the outcome of the first roll. Of the remaining ordered outcomes, half meet the requirement and half do not, because every pair of digits occurs in two arrangements—for example, (4, 1) and (1, 4). We have (36 – 6)/2 = 15, so the probability we need is 15/36 = 5/12.

Alternate solution: Obtain the number of favorable outcomes by observing that if the first roll shows a 1, there are 5 favorable outcomes; if the first roll shows a 2, there are 4 favorable outcomes, …; and if the first roll show a 5, there is only 1 favorable outcome. We have 5 + 4 + 3 + 2 + 1 = 15 favorable outcomes.

4. 5/54. If $a < b < c$, then $2 \le b \le 5$. If $b = 2$, then $a = 1$ and $c \ge 3$, for a total of 4 favorable outcomes. If $b = 3$, then $a \le 2$ and $c \ge 4$, for a total of 2 • 3 = 6 favorable outcomes. If $b = 4$, then $a \le 3$ and $c \ge 5$, for a total of 3 • 2 = 6 favorable outcomes. And if $b = 5$, then $a \le 4$ and $c = 6$, for a total of 4 favorable outcomes. We have 4 + 6 + 6 + 4 = 20 favorable outcomes. There are 6^3 ways to roll a die three times; the desired probability is 20/216 = 5/54.

5. Answers will vary. One possibility follows:

$$\left(\frac{9+5}{7} - 1\right) + 8^3 \cdot 4 - 6^2$$

6. Around 1027 years. Ten sverdrups are 10 million, or 10^7, m^3/sec. Compute the length of time in seconds:

$$\frac{3.24 \cdot 10^{17}\text{ m}^3}{1} \div \frac{10^7\text{ m}^3}{1\text{ sec.}} = \frac{3.24 \cdot 10^{17}\ \cancel{\text{m}^3} \cdot 1\text{ sec.}}{10^7\ \cancel{\text{m}^3}} = 3.24 \cdot 10^{10}\text{ sec.}$$

Convert to years:

$$\frac{3.24 \cdot 10^{10}\ \cancel{\text{sec.}}}{1} \cdot \frac{1\ \cancel{\text{min.}}}{60\ \cancel{\text{sec.}}} \cdot \frac{1\ \cancel{\text{hr.}}}{60\ \cancel{\text{min.}}} \cdot \frac{1\ \cancel{\text{day}}}{24\ \cancel{\text{hr.}}} \cdot \frac{1\text{ yr.}}{365.25\ \cancel{\text{days}}} \approx 1027\text{ yr.}$$

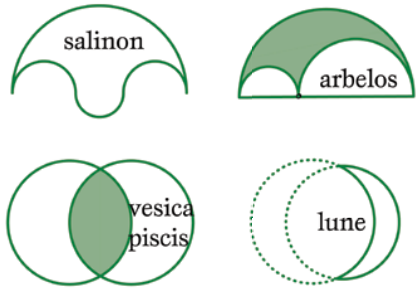
7. $2.92 \cdot 10^6$ times. We have $(3 \cdot 10^9)/1027 \approx 2.92 \cdot 10^6$.

8. –2. Begin by exploring the sequence $a_3 = 2 - 4 = -2$; $a_4 = -2 - 2 = -4$; $a_5 = -4 - (-2) = -2$; $a_6 = -2 - (-4) = 2$; $a_7 = 2 - (-2) = 4$; $a_8 = 4 - 2 = 2$. The 7th and 8th terms are identical to the 1st and 2nd terms, so we know that the entire sequence consists of repetitions of the first 6 terms: 4, 2, –2, –4, –2, 2. Dividing 2013 by 6 leaves a remainder of 3; the 2013th term must be identical to the 3rd term.

9. 6. The prime factors of 2013 are 3, 11, and 61. We have the following:

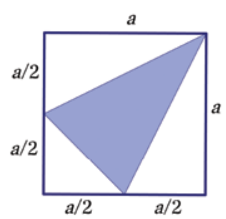
$$\boxed{3} + \boxed{11} + \boxed{61} - \boxed{75} = 5 + 13 + 67 - 79 = 6$$

10. See the figure.



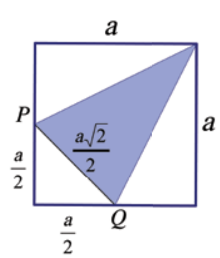
11. $3a^2/8$. Subtract the areas of the three right triangles (unshaded) from the area of the square:

$$a^2 - 2\left(\frac{1}{2}a \cdot \frac{a}{2}\right) - \frac{1}{2}\left(\frac{a}{2}\right)^2$$
$$= a^2 - \frac{a^2}{2} - \frac{a^2}{8}$$
$$= \frac{3a^2}{8}$$



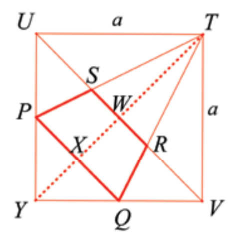
12. $3a\sqrt{2}/4$. Label the endpoints of the base of the isosceles triangle P and Q, as shown. $\overline{PQ}$ is the hypotenuse of an isosceles right triangle; its length is $a\sqrt{2}/2$. We "rearrange" the familiar $A = (1/2)bh$ as $h = 2A/b$, and we use the area of the shaded triangle from the previous problem, $A = 3a^2/8$, to obtain

$$2\left(\frac{3a^2}{8}\right) \cdot \left(\frac{2}{a\sqrt{2}}\right) = \frac{3a\sqrt{2}}{4}.$$



Alternate solution: The altitude is one leg of a right triangle; the other leg is $a\sqrt{2}/4$, and the hypotenuse is $a\sqrt{5}/2$. Use the Pythagorean theorem to find the missing leg.

13. $5a^2/24$. We have the area of $\triangle TPQ$ from January 11: $3a^2/8$. We will find the fraction of that area contained in quadrilateral $PQRS$ without finding the dimensions of $PQRS$. P and Q are midpoints of the sides of $\triangle UVY$, so $\overline{PQ}$ is parallel to $\overline{UV}$. $YQ = QV$, and $YX = XW$, implying that $XW : WT = 1 : 2$ or, equivalently, that $WT : XT = 2 : 3$. Thus, $\triangle TSR$ and $\triangle TPQ$ are similar triangles, with areas in the ratio $4 : 9$, and quadrilateral $PQRS$ has 5/9 the area of $\triangle TPQ$: $(5/9)(3a^2/8) = 5a^2/24$.



14. 24 miles. Since the travelers walk out and back along the same road, the distance uphill and downhill must be the same. Suppose that this distance is 6 miles. The travelers must spend 2 hr. walking up the hill and 1 hr. walking down the hill. Their average rate is then 12 mi./3 hr. = 4 mph. This average rate does not depend on the length of the uphill or downhill stretches. Recognize that this scenario is the classic situation for finding the harmonic mean of two numbers. The harmonic mean of 6 and 3 is

$$\frac{2}{\frac{1}{3} + \frac{1}{6}} = \frac{2}{\frac{3}{6}} = 4.$$

The average of the uphill and downhill rates is the same as the rate on level ground, so the distance traveled is 4 mph • 6 hr. = 24 miles.

Alternate solution: Let x be the distance traveled on level ground in one direction, and let y be the distance traveled uphill or downhill in one direction. Then $x/4 + y/3 + y/6 + x/4 = 6$ hr. Solve for the total distance of $2x + 2y$ by multiplying by 4.

15. $a = 2$, $b = 3$, $c = 5$. Begin by looking for u, the units digit of the common difference. We know that $9 + u$ is a two-digit number with c in the units place and that $9 + 2u$ is a two-digit number with 1 in the units place. The latter relationship implies that $u = 1$ or $u = 6$. Test $u = 1$. Then $c = 0$ and $b = 8$. Substitute these values to obtain $a08$, $a79$, 800, and 881. The first two terms imply that the common difference is 71, but the third and fourth terms contradict that result; thus, the assumption that $u = 1$ must be false. Assume that $u = 6$. Then $c = 5$ and $b = 3$. Substitution yields $a53$, $a79$, 305, and 331. These numbers give a consistent difference of 26; therefore, $a = 2$.

16. \$3.25. Write two equations—one for the actual amount of money that Penny has and one for the amount if the pennies were quarters and the quarters were pennies. Let p, n, d, and q represent, respectively, the number of pennies, nickels, dimes, and quarters that she has. Then $p + 5n + 10d + 25q = 330$, and $25p + 5n + 10d + q = 234$. Subtract the second equation from the first to find $-24p + 24q = 96 \rightarrow q - p = 4$. Penny has four more quarters than pennies. From the first equation, we know that the number of pennies is a nonzero multiple of 5; therefore, the only possibility is 9 quarters and 5 pennies, for a total of \$2.30. The remaining 13 coins are nickels and dimes totaling \$1.00. The solution of the linear system $n + d = 13$ and $5n + 10d = 100$ is $d = 7$, $n = 6$. Reversing the numbers for nickels and dimes, we have 6 dimes and 7 nickels. If the purse contained 9 quarters, 6 dimes, 7 nickels, and 5 pennies, Penny would have \$3.25.

17. 15. The height of the hollow cylinder is given by point B's z-coordinate—25. The cylinder is hollow, so subtract the volume of the cylindrical "hole" from the volume of a solid cylinder with height 25 and radius k. We have $V = 25\pi k^2 - 25\pi \cdot 10^2$. When the rectangle rotates about the y-axis, it creates a cylinder with height $k - 10$ and radius 25. Its volume is given by $V = 625(k - 10) \cdot \pi$. Since the two volumes are equal, we have the following:

$$25\pi k^2 - 25\pi \cdot 10^2 = 625(k - 10)\pi \rightarrow$$
$$25\pi(k^2 - 100) = 625(k - 10)\pi \rightarrow$$
$$k^2 - 25k + 150 = 0 \rightarrow$$
$$k = 10 \text{ or } k = 15$$

Reject $k = 10$.

18. 15 6/7 coins (dog), 31 5/7 coins (wild cat), 63 3/7 coins (fox). Let d = the amount that the dog's skin is worth. Then the wild cat's skin is worth $2d$, and the fox's skin is worth $4d$. We have $7d = 111 \rightarrow d = 15\ 6/7$ coins. The dog must pay 15 6/7 coins, the wild cat must pay 31 5/7 coins, and the fox must pay 63 3/7 coins. (The fractional parts of coins are consistent with the solution in the original text.) (Source: Cited in Jacqueline Stedall, *The History of Mathematics: A Very Short Introduction*, Oxford University Press, 2012, p. 20.)

19. 2^{60}. The least common multiple of 2, 3, 4, 5, and 6 is 60. Thus,

$$x^{60} = \left(x^{30}\right)^2 = \left(x^{20}\right)^3 = \left(x^{15}\right)^4 = \left(x^{12}\right)^5 = \left(x^{10}\right)^6$$

The smallest such integer greater than 1 is 2^{60}.

20. 3D • D 0 = 2C • I 5 = 2790. One approach converts 2790 to base 10: $2(20^3) + 7(20^2) + 9(20) = 16{,}000 + 2{,}800 + 180 = 18{,}980$. Then $3D = 3(20) + 13 = 73$, and $18{,}980/73 = 260$. Since $260_{10} = 20 \cdot 13$, the first two blanks in the equation must be D0. The two-digit number with 2 in the twenties place can be any integer between 40 and 59, inclusive. Only one integer in this interval is a factor of 18,980: 52. We have $52_{10} = 2C_{20}$. The final integer is 18980/52 = 365, which becomes I5 in vigesimal notation.

The Maya used a vigesimal system. They had two calendars—a ritual calendar (with 260 days) and a civil calendar (with 365 days). The two calendars would return to the same cycle after 18,980 days. This fact explains the particular equation used in this problem. (See the article by Gray and Rice in this issue, pp. 338–44.)

21. 15. We know that dividing by 9 + 9 = 18 allows the largest possible remainder: 17. Can we find a case in which the remainder actually is 17? Since 99/18 results in a remainder of 9, we know that we cannot obtain a remainder of 17. We might have a remainder of 16 if we divide by 17. We have either 89/17 or 98/17. The former gives us a remainder of 4; the latter gives us a remainder of 13. Can we find a larger remainder? We try dividing by 16 and find that 79/16 = 4 with remainder 15.

22. 33, 34, 35. The square-free semiprimes between 14 and 35 are $3 \cdot 5 = 15$; $3 \cdot 7 = 21$; $2 \cdot 11 = 22$; $2 \cdot 13 = 26$; $3 \cdot 11 = 33$; $2 \cdot 17 = 34$; and $5 \cdot 7 = 35$. Note that there can never be four consecutive integers that are all square-free semiprimes because in any set of four consecutive integers, there is one that is divisible by $4 = 2^2$.

23. Four ways. The nonconvex hexagon *ABCDEF* has area 5 square units. To obtain area 6 units^2, point B can be moved 1 unit to the left. An area of 6 results if point C is moved up 1 unit or left 1 unit. Moving point D 1 unit to the right also increases the area by 1.

24. 358. The largest product that can be obtained from a two-digit number is $9 \cdot 9 = 81$, so we will look for a three-digit integer whose digits have the product 120. This integer cannot have 1 in the hundreds place because the remaining two digits would have to have a product of 120. If the integer has 2 in the hundreds place, the remaining two digits would have to have a product of 60. The largest single-digit factor of 60 is 6, which pairs with 10 for a product of 60. If the integer has 3 in the hundreds place, the remaining two digits would have to have a product of 40. Since $5 \cdot 8 = 40$, the integer we seek is 358.

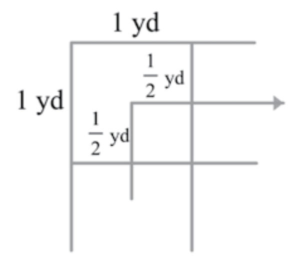
Alternate solution: The largest single-digit factor of 120 is 8. Place that digit in the ones place. The remaining digits must have a product of 15. The largest single-digit factor is 5. Place that digit in the tens place. The remaining digit is 3.

25. $10^2 + 20^2 + 27^2 + 28^2$; $19^2 + 20^2 + 24^2 + 26^2$; or $5^2 + 8^2 + 18^2 + 40^2$. Although 2013 cannot be written as the sum of two squares, it can be written as the sum of three squares: $2^2 + 28^2 + 35^2$.

26. 5/18. There are $6 \cdot 6 = 36$ possible outcomes, of which 10 are perfect squares, as shown in the table. The probability of a perfect square is thus 10/36 = 5/18. (See diagram, top col. 3.)

	6	10	15	21	28	36
6	12	16	21	27	34	42
10	16	20	25	31	38	46
15	21	25	30	36	43	51
21	27	31	36	42	49	57
28	34	38	43	49	56	64
36	42	46	51	57	64	72

27. 60 × 60.5 yards. Let a walker stand at the beginning of the spiral path. Imagine that he begins to traverse the entire gravel-walk. Since the path is 1 yard wide, he can count off 1 square yard for each linear yard he walks. This one-to-one correspondence applies to corners as well as to the straight sections:



Thus, the area of the garden is 3630 square yards. Let y = the shorter dimension of the garden. Then we have the following:

$$y(y + 1/2) = 3630$$
$$y^2 + (1/2)y = 3630$$
$$(y + 1/4)^2 = 3630 + 1/16$$
$$\left|y + \frac{1}{4}\right| = \sqrt{\frac{58081}{16}}$$
$$y = -1/4 + 241/4$$
$$y = 60 \text{ yards}$$

The longer side is 60.5 yards.

Alternate solution: Imagine that we can unwind the spiral to obtain a rectangle measuring 1 yard × 3630 yards. What other rectangles have the same area? Specifically, can we find a rectangle that is "almost" a square? We know that $60^2 = 3600$ and $3630 = 60^2 + (1/2)(60)$. So the dimensions 60 × 60.5 yards give us what we seek.

28. 48. If we begin with a diagonal move from C to A (4 possibilities), then the following move must be either

vertical or horizontal (2 possibilities). The next two moves, to R and then to O, allow no choices. The two diagrams show that there are 4 ways to complete the name once we reach an O:

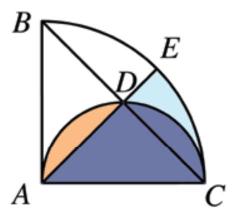
$$\begin{array}{ccccc} O & \rightarrow & L & \rightarrow & L \\ \downarrow & & & & \\ L & & & & \\ \downarrow & & & & \\ L & & & & \end{array} \qquad \text{or} \qquad \begin{array}{ccc} O & \rightarrow & L \\ \downarrow & \swarrow\!\!\nearrow & \\ L & & \end{array}$$

So we have 4 • 2 • 4 = 32 paths. Suppose, on the other hand, that the first move, from C to A, is vertical or horizontal (4 possibilities). Then there is no choice for the second or third or fourth move. There are 4 ways to complete the name once we reach an O, as before. We have found 4 • 4 = 16 additional paths, for a total of 48 paths.

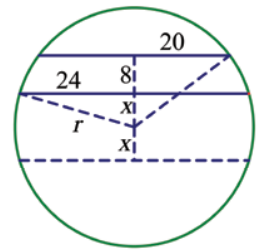
29. 1 : 1. It suffices to show that sector EAC has the same area as the semicircle with diameter AC to prove that the regions shaded blue and tan have equal areas. Let $AC = r$. Then the radius of semicircle $\widehat{ADC} = r/2$, and the area of the semicircle is

$$\frac{1}{2}\pi\left(\frac{r}{2}\right)^2 = \frac{\pi r^2}{8}.$$

We know that $m\angle ACB = 45°$ and that $m\angle ADC = 90°$ since $\angle ADC$ is inscribed in a semicircle. Thus, $m\angle DAC = 45°$, and the area of sector EAC is $\pi r^2/8$, the same area as that of the semicircle.



30. 22 units. The radius perpendicular to the two chords bisects them. Half of each chord is a leg of a right triangle with hypotenuse r, the radius of the circle. Let the remaining legs be x and $x + 8$, as shown. Then $x^2 + 24^2 = r^2$, and $(x + 8)^2 + 20^2 = r^2$. Solve for x: $x^2 + 24^2 = (x + 8)^2 + 20^2 \rightarrow x = 7$. If the center of the circle lies between the two chords, the distance between them is 8 + 7 + 7 = 22 units. (Although we have enough information to find the radius of this circle, we are able to answer the question without doing so.)



31. 25. The fraction shaded in the first figure is 1, in the second figure 3/4, and in the third figure 6/9 = 2/3. Extend the pattern: 10/16, 15/25, 21/36, and so on. We observe that the numerators of the (unreduced) fractions are the triangular numbers and that the denominators are the square numbers. Written as a fraction, 52% = 52/100. Although 100 is a perfect square, 52 is not a triangular number. Written as a reduced fraction, 52% = 13/25. Again, 13 is not a triangular number, so we look for perfect squares that are divisible by 25: $5^2 = 25$, $15^2 = 225$, and $25^2 = 625$, to begin with. The 5th triangular number = 15, the 15th triangular number = 120, and the 25th triangular number = 325. The fraction 325/625 is equivalent to 13/25, so $n = 25$.

Alternate solution: After observing the pattern of triangular and square numbers, use the formulas for each to write and solve the equation:

$$\frac{\frac{n(n+1)}{2}}{n^2} = 0.52$$
$$\frac{n(n+1)}{2n^2} = 0.52$$
$$\frac{n+1}{2n} = 0.52$$
$$n = 25$$

FEBRUARY

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Theme: For each problem in which a mathematician is mentioned, the answer is related to the birth year of that mathematician.

A Pythagorean quadruple satisfies the equation

$$a^2 + b^2 + c^2 = d^2$$

where a, b, c, and d are positive integers. If three of the values of a quadruple are 272, 306, and 408, determine the fourth value.

1

Zeno ran toward a stationary bus. In the first time interval, he covered 247.5 cm. In each successive time interval, he covered half the distance covered in the previous time interval. He did this for an infinite number of tiny time intervals and finally caught up to the bus. How many centimeters away was the bus?

2

Euclid claimed that if p is a prime number and p is a factor of the product ab, then p must be a factor of a or b. Assuming that this statement is true and that $p = 23$ when $a = 14$, find the least possible value of ab.

3

Archimedes proved that the area between a line and a parabola equals 4/3 the area of the inscribed triangle formed by the two points of intersection and the point on the parabola midway between the x-values of those intersections. Find the area between the parabola defined by $64y = 144 - x^2$ and the line defined by $8y = 3x - 36$.

4

A circle of Apollonius is a circle tangent to three given circles. When each of the three given circles is tangent to the other two, two Apollonian circles exist. If the radii of the three given circles are 50, 80, and 160, find the radius of the outermost circle to the nearest whole number. Use the following formula:

$$r = \left| \frac{r_1 r_2 r_3}{r_1 r_2 + r_2 r_3 + r_3 r_1 - 2\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}} \right|$$

5

Fibonacci was preparing for a marathon by running up a set of 15 steps. He decided to run up the stairs in every possible way—by going up 1 step at a time, going up 2 steps at a time, and using every combination of 1 and 2 steps. Unfortunately, the total number of ways fell short of the year of his birth by 183. In what year was Fibonacci born?

6

A formula for solving cubic polynomials was first published by Cardano in 1545. His solutions for a given cubic follow:

$$x_1 = \sqrt[3]{857 + 31i} + \sqrt[3]{857 - 31i},$$

$$x_2 = \sqrt[3]{61610 + 1913i} + \sqrt[3]{61610 - 1913i},$$

and $x_3 = 1$

Compute the constant term of the cubic to the nearest integer. Use a CAS calculator or computer program or the website www.wolframalpha.com for assistance.

7

Since $10^3 = 1000$ and $10^4 = 10{,}000$, $10^{3.1903317}$ must be between 1000 and 10,000. John Napier found a way to calculate this value. What is the value?

8

The development of analytic geometry is credited to Rene Descartes along with the current use of superscripts to indicate exponential operations. Use the laws of exponents to evaluate this expression without using a calculator:

$$2^6 \cdot 5^2 - 2^2$$

9

One theorem attributed to Fermat states that every odd prime of the form $4k + 1$ can be expressed as the sum of two perfect squares. For example, $5 = 1^2 + 2^2$ and $29 = 2^2 + 5^2$. Research the year of Fermat's birth and then find the two squares whose sum is that year.

10

The sum of numbers in the first ten rows of Pascal's triangle differs from Pascal's birth year by a number that factors into $2^3 \cdot 3 \cdot 5^2$. Determine the year that he was born.

11

Newton approximated the solution of $x^2 - 10x - 105 = 0$ by making a guess called x_1 and substituting it into the equation

$$x_2 = \frac{x_1 - \left(x_1^2 - 10x_1 - 105\right)}{2x_1 - 10}$$

to find the next estimate of the solution. He would then repeat the same step with x_2. If $x_1 = 17$, find x_2 to the nearest hundredth. Multiply this by 100 to find Newton's birth year.

12

To solve two equations in two unknowns, Leibniz reduced the matrix of coefficients to the identity (unit) matrix by using elementary row operations. Research Leibniz's method and use it to solve the following system of equations:

$$5x - 2y = -12 \text{ and } -7x + 3y = 26$$

The resulting solution when concatenated is Leibniz's birth year.

13

Jacob Bernoulli experimented with laws of probability. His younger brother, Johann, was born in the year that was 10,000 times the probability of getting an even prime number when rolling one die one time (rounded to the nearest integer). What year was Johann born?

14

Euler's number, $e \approx 2.718281828459045\ldots$, can be found to many decimal places in a variety of ways. If we ignore the first 10,394 digits in the number, the next four will be the year that Euler was born. Find that year.

15

Gauss proved that every positive integer is the sum of at most three triangular numbers. Verify that this statement is true for both the year that he was born (1777) and the year that he died (1855).

16

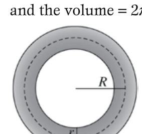
Adding 1 to the sum of the squares of the integers from −6 to 11 produces the year of Brahmagupta's birth. Find it.

17

Dirichlet was the first to formalize the pigeonhole principle. His idea may be used to solve this problem: A bag of jelly beans contains 335 red, 357 blue, 592 white, 294 orange, and 520 green beans. What is the minimum number of jelly beans that must be selected to guarantee at least one bean of every color. (No peeking.)

18

Pappus discovered formulas for the surface area and the volume of a torus (a donut-shaped object). The surface area $= 4\pi^2 rR$, and the volume $= 2\pi^2 r^2 R$. The sum of the surface area and the volume can be expressed as $k\pi^2$. If $r = 3$ and $R = 10$, find k.



19

Riemann found a technique for calculating the area between the x-axis and the parabola $y = x^2$. He found that to determine the area from a to b, one must compute $(b^3 - a^3)/3$. Find the year that Riemann was born by subtracting 11/3 from the area under $y = x^2$ from $x = 7$ to $x = 18$.

20

Frobenius asked, "What is the largest integer that cannot be expressed as the sum of multiples of a given set of integers?" McNuggets® are sold in packs of 6, 9, and 20. Square the largest integer that is not the sum of multiples of these three integers to produce the year that Frobenius was born.

21

The Markov equation

$$x^2 + y^2 + z^2 = 3xyz$$

has many solutions. If $x = 5$, there are still an infinite number of solutions, including $y = 1$ and $z = 2$ and one other solution with $y = 2$ and z a prime number. Calculate $y^6 z$ to find the year that Markov was born.

22

Giuseppe Peano published a book on mathematical logic that introduced the symbols now used for union and intersection. The number of elements in a set S is written as $n(S)$, and the complement of set S is written $\sim S$. If $n(A \cap \sim B) = 1123$, $n(B \cap \sim A) = 400$, and $n(A \cap B) = 335$, find $n(A \cup B)$.

23

Hilbert numbers are positive integers of the form $4n + 1$. Find the sum of the Hilbert numbers from 17 to 121. Subtracting 1 from your result will show the year that David Hilbert was born.

24

The Hausdorff-Besicovitch dimension formula can be used to calculate the dimension of a fractal. Lines are one dimensional, whereas squares are two dimensional. The dimension of a wiggly line falls somewhere between one and two. Use the definition $d = \log(n)/\log(r)$ to approximate $1000d$ to the nearest unit when n, the number of wiggles, is 505 and r, the ratio of original length to new length, is 28.

25

Hardy posed several conjectures concerning types of prime numbers. Oddly, he was born in a prime-numbered year that was the sum of the squares of three consecutive integers. The two smallest primes with this property are 29 $(2^2 + 3^2 + 4^2)$ and 509 $(12^2 + 13^2 + 14^2)$. Hardy was born in the smallest four-digit prime year with this property. Find this year.

26

Benoit Mandelbrot studied iteration with complex numbers and the function $f(x) = x^2 + c$ where c is a constant. When $c = 0.2115 + 0.1488i$ and the seed (initial) value for x is $0 + 0i$, the fixed point after about 40 iterations can be written as $p + qi$. Compute $10{,}000p + 100q$. The result is the year that Mandelbrot was born.

27

John Conway developed a base-13 function that was discontinuous at every point. Use the letters A, B, and C to represent the digits for the numbers ten, eleven, and twelve and then determine the year that Conway was born if in base 13 the year is B60.

28

SOLUTIONS to calendar

FEBRUARY

Looking for more Calendar problems?
Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from *Mathematics Teacher.*

*Edited by **Margaret Coffey,** Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312, and **Art Kalish,** artkalish@verizon.net, Syosset High School (retired), Syosset, NY 11791*

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

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1. 578. Either $d = 408$, or we seek d. If we let d be 408, we obtain $272^2 + 306^2 + c^2 = 408^2$. Then, $c^2 = -1156$, a result that is not possible. Thus, $d^2 = 272^2 + 306^2 + 408^2 = 334084$, and $d = 578$.

2. 495. The sum of an infinite geometric series is given by $S = a/(1 - r)$ where a is the first term, or 247.5, and r is the ratio of successive terms, or 0.5. Thus, $S = 247.5/(1 - 0.5) = 495$.

3. 322. The least value for b is 23, so the least value for $ab = 14(23) = 322$.

4. 288. Find the coordinates of the intersections of the line and the parabola. Since $8y = 3x - 36$, $64y = 24x - 288$. We know that $64y = 144 - x^2$, so $24x - 288 = 144 - x^2 \rightarrow x^2 + 24x - 432 = 0$. Solve to get $x = 12$ or $x = -36$. Substitute these values into the linear equation to find the corresponding y-values. The intersections are $(12, 0)$ and $(-36, -18)$. The x-value midway between these x-values is $(12 + -36)/2 = -12$, and the point on the parabola with this x-coordinate is $(-12, 0)$. The base of the triangle is 24, and its height is 18, so the area $= (0.5) \cdot (24)(18) = 216$; 4/3 of this result is 288.

5. 262. This problem requires careful substitution into a calculator. The radius of the inner circle tangent to the three given circles can be found by changing the subtraction sign in the denominator to an addition sign.

6. 1170. To find the number of ways to climb a set of 15 stairs by going 1 step at a time, going 2 steps at a time, or using any combination of 1 and 2 steps, simplify the problem. Examine the number of ways to go up just 1 step (1 way); 2 steps (2 ways: 1 + 1 or 2); 3 steps (3 ways: 1, 1, 1 or 1, 2 or 2, 1); and 4 steps (5 ways: 1, 1, 1, 1 or 1, 1, 2 or 1, 2, 1 or 2, 1, 1 or 2, 2). You may realize that to climb 5 steps, you must first go up either 1 or 2 steps. If you go up 1 step, there are 4 remaining steps, and we know that there are 5 ways to finish the task. If you initially go up 2 steps, there are 3 remaining steps, and we know that there are 3 ways to finish the task. Thus, there are 5 + 3, or 8, ways to go up 5 stairs. Extend this pattern: To go up 6 stairs, add 8 + 5 = 13 ways (you must initially go up 1 or 2 stairs, leaving the two previous conditions as the remaining number of steps). Continue the pattern for 15 stairs: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, and 987. To find the year that Fibonacci was born, add 183 to get 1170.

7. 1501. If you do not have a calculator that can handle this problem, go to the website mentioned in the problem. The constant term of the cubic is the opposite of the product of the three given solutions.

8. 1550. Most calculators can perform this operation. What do you think was done to find the exponent in this question so that the result would be the year that Napier was born?

9. 1596. The easiest way to perform this operation is to rewrite the expression using the principle $a^m b^m = (ab)^m$. So we can write $2^6 \cdot 5^2 - 2^2 = 2^2 \cdot 5^2 \cdot 2^4 - 2^2 = 10^2 \cdot 16 - 4 = 1600 - 4 = 1596$.

10. 1601, 40, and 1. Fermat was born in 1601, so we are looking for the sum of two squares with that sum. This answer is unique, as are all sums found in this way.

11. 1623. Each row in Pascal's triangle starts and ends with the number 1. Interior elements are formed by adding the two numbers immediately above. Thus, the next row in the triangle shown in the

figure will begin 1, 1 + 6 = 7, 6 + 15 = 21, 15 + 20 = 35, and so on. To find the sum of the first 10 rows, we can determine all ten rows and add all the numbers. Alternatively, we can find the sum of each row. The sums of the first five rows are 1, 2, 4, 8, and 16. Notice that each row has a sum that is twice the sum of the preceding row. This pattern will continue for all ten rows, forming a finite geometric sequence. We want the sum of these rows, so we apply the formula $S = (a - ar^n)/(1 - r) = (1 - 1 \cdot 2^{10})/(1 - 2) = 1023$. If we add to this result the product of the factors $2^3 \cdot 3 \cdot 5^2 = 600$, we get 1623.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

12. 1642. Substitute $x_1 = 17$ into the formula:

$$\begin{aligned} x_2 &= x_1 - \frac{x_1^2 - 10x_1 - 105}{2x_1 - 10} \\ &= 17 - \frac{17^2 - 10(17) - 105}{2(17) - 10} \\ &\approx 16.42 \end{aligned}$$

Multiplying this result by 100 and rounding to the nearest integer gives 1642.

13. 1646. When using matrices to solve this system of equations, we would form the initial matrix as

$$\left|\begin{array}{ccc} 5 & -2 & -12 \\ -7 & 3 & 26 \end{array}\right|.$$

Then, using multiplication and addition of rows, we would try to get the result

$$\left|\begin{array}{ccc} 1 & 0 & a \\ 0 & 1 & b \end{array}\right|$$

where (a, b) is the point common to the two equations. To begin, multiply the top row by 3 and the bottom row by 2:

$$\left|\begin{array}{ccc} 5 & -2 & -12 \\ -7 & 3 & 26 \end{array}\right| \rightarrow \left|\begin{array}{ccc} 15 & -6 & -36 \\ -14 & 6 & 52 \end{array}\right|$$

Add the two rows and put the sum in the top row:

$$\left|\begin{array}{ccc} 1 & 0 & 16 \\ -14 & 6 & 52 \end{array}\right|$$

Multiply the top row by 14 and add it to the bottom row:

$$\left|\begin{array}{ccc} 1 & 0 & 16 \\ 0 & 6 & 276 \end{array}\right|$$

Finally, divide the bottom row by 6:

$$\left|\begin{array}{ccc} 1 & 0 & 16 \\ 0 & 1 & 46 \end{array}\right|$$

These manipulations give us the solution (16, 46). If we concatenate these two numbers, we get 1646, or the impossible date 4616.

14. 1667. The only even prime is 2, and the probability of rolling a 2 is 1/6. Multiply this by 10,000 to get 1666.667 → 1667.

15. 1707. A website that gives the first two million digits of e is http://apod.nasa.gov/htmltest/gifcity/e.2mil. Copy a portion of the number into Microsoft Word and then use the **Word Count** option to find the 10,393rd digit. The next four digits are Euler's birth year. Be careful to exclude the decimal point in your count.

16. 1777 = 1 + 6 + 1770 and 1855 = 10 + 15 + 1830. The triangular numbers are the set of numbers formed by adding the natural numbers. The first six numbers are 1, 3, 6, 10, 15, and 21. The formula that can be used to generate these numbers is $T_n = 1 + 2 + 3 + \cdots + n = n(n + 1)/2$. Make a list of these numbers in a program such as Excel® and find the three numbers that sum to the required years.

17. 598. One way to add the squares of these integers is to apply the formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

twice. For the numbers –6 to –1, the sum is 6(7)(13)/6 = 91; and for the numbers 1 to 11, the sum is 11(12)(23)/6 = 506. Add 1 to the sum of these two results to get 91 + 506 + 1 = 598.

18. 1805. Apply the worst-case scenario to assume that all the red, blue, white, and green jelly beans are selected first. That total is 335 + 357 + 592 + 520 = 1804. One more bean must be selected to ensure that an orange bean is included, so the total is 1805.

19. 300. Substitute the values $r = 3$ and $R = 10$ into the formulas for surface area and volume. The surface area $= 4\pi^2(3)(10) = 120\pi^2$, and the volume $= 2\pi^2 r^2 R = 2\pi^2(3^2) \cdot (10) = 180\pi^2$. Thus, the sum is $300\pi^2$.

20. 1826. Since $a = 7$ and $b = 18$, $A = (18^3 - 7^3)/3 = 5489/3$. After subtracting 11/3, we have 5478/3 = 1826.

21. 1849. Make a list of the counting numbers and examine which numbers are the result of adding multiples of 6, 9, and 20 (see the chart below). It should be obvious that none of the numbers less than 6 can be created. By adding 6 to

6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
Y	N	N	Y	N	N	Y	N	N	Y	N	N	Y	N	Y	Y	N	N	Y	N	Y

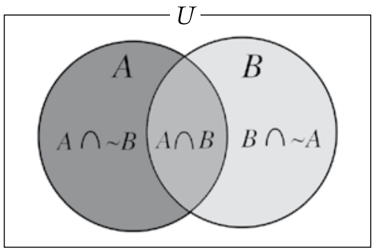
27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
Y	N	Y	Y	N	Y	Y	N	Y	Y	N	Y	Y	Y	Y	Y	N	Y	Y	Y

47	48	49
Y	Y	Y

each of the last six entries, we can form the next six entries, and so on. Therefore, the largest number of McNuggets that cannot be formed is 43, and $43^2 = 1849$.

22. 1856. In the equation $x^2 + y^2 + z^2 = 3xyz$, replace x with 5 and y with 2; then $25 + 4 + z^2 = 30z$. Solve the equation $z^2 - 30z + 29 = 0 \rightarrow (z - 29)(z - 1) = 0$; so $z = 29$ or $z = 1$. Since 1 is not prime, $z = 29$, and $y^6 z = 2^6 \cdot 29 = 1856$.

23. 1858. If a picture is worth a thousand words, here is a Venn diagram that we can use to answer this problem. The union of A and B is the sum of the three shaded regions: $1123 + 400 + 335 = 1858$.



24. 1862. The Hilbert sequence is an arithmetic sequence. To find the sum of this particular sequence, use $a_n = a_1 + (n - 1)d$ to find n, the number of terms, and then $S_n = n(a_1 + a_n)/2$ to compute the sum. Then we have $121 = 17 + (n - 1)4 \rightarrow n = 27$; $S_n = 27(17 + 121)/2 = 1863$; and $1863 - 1 = 1862$.

25. 1868. Substitute $n = 505$ and $r = 28$ into the formula $d = \log(n)/\log(r) = \log(505)/\log(28) = 1.868000$. Then multiply the result by 1000 to get 1868.

26. 1877. The first set of consecutive square integers with a four-digit sum is $18^2 + 19^2 + 20^2 = 1085$, which is not prime. When we start with an odd integer, the sum will always be even and, hence, not a prime. Try starting with 20, 22, and 24. The first triple to produce a prime result is (24, 25, 26). That is, $24^2 + 25^2 + 26^2 = 1877$, the year that Hardy was born.

27. 1924. You will need a calculator or Excel® to solve this problem. If your calculator can work with complex numbers, type **0** and then **enter**. This action stores the number 0 in a place called **ans**. Next, type **ans^2+.2115+.1488i** and then hit **enter**. Continually hitting **enter** will take the current result and place it in **ans**. Repeat until all changes appear to stop. The result is a fixed point for this function—that is, $f(x) = x$. The number is $0.19 + 0.24i$, and $10{,}000(0.19) + 100(0.24) = 1924$.

28. 1937. To convert B60 into base 10, simplify $\mathrm{B}(13^2) + 6(13) + 0(1) = 11(169) + 6(13) = 1937$.